Proceedings of IAM, V.15, N.1, 2026, pp.94-104
https://doi.org/10.30546/2225-0530.15.01.2026.094

MULTI-SUBLINEAR MAXIMAL COMMUTATORS ON PRODUCT MODIFIED MORREY SPACES

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Abstract: In this paper, we obtain necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator operatör $\mathcal{M}_{\vec{b}}$ from the product modified Morrey spaces $\widetilde{L}^{p_1,\lambda_1}(R^n)\times\ldots\times\widetilde{L}^{p_m,\lambda_m}(R^n)$ to modified Morrey spaces $\widetilde{L}^{p,\lambda}(R^n)$.

Keywords: Modified Morrey spaces, multi-sublinear maximal function, multi-sublinear maximal commutator, BMO.

AMS Subject Classification: 42B20, 42B25, 42B35.

1. Introduction

Morrey spaces, introduced by Morrey [25], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. However, in the theory of partial differential equations, along with Morrey spaces, modified Morrey spaces also play an important role [2, 3, 4, 7, 11, 13, 14, 22, 23, 24, 26, 28, 29]. The modified Morrey spaces $\widetilde{L}^{p,\lambda}(R^n)$, introduced by V.P. Il'in [19], consist of all locally integrable functions f on R^n such that the following finite (quasi-)norm is finite

$$\|f\|_{\widetilde{L}^{p,\lambda}(R^n)}=\sup_{x\in R^n,t>0}[t]_1^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at $x\in R^n$ with radius $t>0$, $[t]_1=\min\{1,t\}$, see [7, 8].

Let $(R^n)^m=R^n\times\ldots\times R^n$ be the m-fold product space $(m\in N)$. Form $x\in R^n$ and $r>0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{C}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\vec{f}$ the m-tuple $(f_1,f_2,\ldots,f_m)$, $\vec{y}=(y_1,y_2,\ldots,y_m)$ and $d\vec{y}=dy_1\ldots dy_m$.

In the past twenty years, the multilinear Calderon-Zygmund theory was developed a lot and studied by many authors. Grafakos and Torres [6] introduced the multilinear Calderon-Zygmund operator and studied the boundedness of such operators. Later in [21], Lerner et al. introduced the following multi-sublinear maximal function $\mathcal{M}\left(\vec{f}\right)(x)$ defined as

$$\mathcal{M}\left(\vec{f}\right)(x)=\sup_{B\ni x}\prod_{i=1}^{m}\frac{1}{|B|}\int_B |f_i(y_i)|\,dy_i .$$

When $m=1$, we get $M\equiv M_1$ the classical Hardy-Littlewood maximal operator is given by

$$M\ f(x)=\sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|\,dy,$$

where f is a locally integrable function. Obviously, it is easy to see $\mathcal{M}\left(\vec{f}\right)(x)\leq \prod_{j=1}^{m}M\ f_j(x).$

The sharp maximal function of Fefferman and Stein $M^{\neq} f$ is defined by

$$M^{\neq} f(x)=\sup_{B\ni x}|B|^{-1}\int_B |f(y)-f_B|\,dy,$$

where the supremum is taken over all balls $B\subset R^n$ containing x. For a fixed $q\in(0,1)$, any suitable function h and $x\in R^n$, let $M_q^{\neq}h(x)=\left(M^{\neq}\left(|h|^q\right)(x)\right)^{1/q}$ and $M_q h(x)=\left(M\left(|h|^q\right)(x)\right)^{1/q}$.

For any ball $B\subset R^n$, the mean oscillation space $BMO(R^n)$ is defined as

$$BMO(R^n)=\left\{b\in L_{loc}(R^n):\|f\|_{BMO}:=\sup_B |B|^{-1}\int_B |f(y)-f_B|dy<\infty\right\},$$

where $f_B=|B|^{-1}\int_B f(y)dy$.

In [9], Guliyev give the necessary and sufficient conditions for the boundedness of the maximal commutator operator

$$M_b f(x)=\sup_{B\ni x}|B|^{-1}\int_B |b(x)-b(y)||f(y)|dy,$$

and the commutator of Hardy-Littlewood maximal operator

$$[b,M]f(x)=b(x)Mf(x)-M(bf)(x)$$

on $\widetilde{L}^{p,\lambda}(R^n)$, where $b\in BMO(R^n)$ (see also [10]).

The commutators of singular integral operators plays an important role in studying the regularity of solutions of elliptic partial differential equations. In [31, 32] Zhang studied the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ and the commutator of multi-sublinear maximal operator $[\vec{b},\mathcal{M}]$ generated by the BMO^m vector function defined as follows:

$$\mathcal{M}_{\vec{b}}\left(\vec{f}\right)(x) = \sum_{j=1}^{m} \mathcal{M}_{b_j}\left(\vec{f}\right)(x) \text{ and } [\vec{b},\mathcal{M}]\left(\vec{f}\right)(x) = \sum_{j=1}^{m} [\vec{b},\mathcal{M}]_j \left(\vec{f}\right)(x).$$

Here

$$\mathcal{M}_{b_j}\left(\vec{f}\right)(x) = \sup_{B\ni x} \frac{1}{|B|^m} \int_B |b_j(x) - b_j(y_j)| \prod_{i=1}^{m} |f_i(y_i)| d\vec{y}$$

and

$$[\vec{b},\mathcal{M}]_j(\vec{f})(x) = b_j(x)\mathcal{M}\left(\vec{f}\right)(x) - \mathcal{M}\left(f_1,\ldots,f_{j-1},b_j f_j, f_{j+1},\ldots,f_m\right)(x).$$

Obviously, the $\mathcal{M}_{\vec{b}}$ and $[\vec{b},\mathcal{M}]$ operators are essentially different from each other because $\mathcal{M}_{\vec{b}}$ is positive and sublinear and $[\vec{b},\mathcal{M}]$ is neither positive nor sublinear.

In this paper, we obtain necessary and sufficient conditions for the boundedness of multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ on product modified Morrey spaces $\widetilde{L}^{p_1,\lambda}(R^n) \times \ldots \times \widetilde{L}^{p_m,\lambda}(R^n)$.

This paper is organized as follows: In Section 2, we give some theorems about the boundedness of multi-sublinear maximal operator $\mathcal{M}$ on the product modified Morrey spaces $\widetilde{L}^{p,\lambda}(R^n)$. In Section 3 we find necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ from product modified Morrey space $\widetilde{L}^{p_1,\lambda}(R^n) \times \ldots \times \widetilde{L}^{p_m,\lambda}(R^n)$ to modified Morrey spaces $\widetilde{L}^{p,\lambda}(R^n)$, see also [11, 12, 13, 17, 20, 23, 27, 28, 29, 30].

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Multi-sublinear maximal operator on product modified Morrey spaces

In this section, we investigate the boundedness of multi-sublinear maximal operator $\mathcal{M}$ on product modified Morrey spaces.

Definition 1. *Let* $0 < p < \infty$, $\lambda \in R$, $[t]_1 = \min\{1,t\}$, $t > 0$. *We denote by* $L^{p,\lambda}(R^n)$ *the classical Morrey space, and by* $\widetilde{L}^{p,\lambda}(R^n)$ *the modified Morrey space [7] the set of all classes of locally integrable functions* f *with the finite quasi-norms*

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in R^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))}, \qquad \|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))},$$

respectively.

Definition 2. *Let* $0<p<\infty$, $\lambda\in R$. *We define the weak Morrey space* $WL^{p,\lambda}\left(R^n\right)$, *and the weak modified Morrey space* $W\widetilde{L}^{p,\lambda}\left(R^n\right)$ *[7] as the set of all locally integrable functions* f *with finite quasi-norms*

$$\|f\|_{WL^{p,\lambda}}=\sup_{x\in R^n,t>0} t^{-\frac{\lambda}{p}}\|f\|_{WL^p(B(x,t))}, \qquad \|f\|_{W\widetilde{L}^{p,\lambda}}=\sup_{x\in R^n,t>0} t_1^{-\frac{\lambda}{p}}\|f\|_{WL^p(B(x,t))},$$

respectively.

Lemma 1. *[9, 10] If* $0<p<\infty$, $0\le\lambda\le n$, *, then*

$$\widetilde{L}^{p,\lambda}\left(R^n\right)=L^{p,\lambda}\left(R^n\right)\cap L^p\left(R^n\right),$$
$$W\widetilde{L}^{p,\lambda}\left(R^n\right)=WL^{p,\lambda}\left(R^n\right)\cap WL^p\left(R^n\right)$$

and

$$\|f\|_{\widetilde{L}^{p,\lambda}\left(R^n\right)}=\max\left\{\|f\|_{L^{p,\lambda}},\|f\|_{L^p}\right\},$$
$$\|f\|_{W\widetilde{L}^{p,\lambda}\left(R^n\right)}=\max\left\{\|f\|_{WL^{p,\lambda}},\|f\|_{WL^p}\right\},$$

respectively.

The following local estimate is valid.

Lemma 2. *[1, Lemma 3.3] Let* $1\le p<\infty$ *and* $B(x,r)$ *be any ball in* R^n. *Then, for* $p>1$ *the inequality*

$$\|Mf\|_{L^p(B(x,r))}\lesssim r^{\frac{n}{p}}\sup_{t>2r}t^{-\frac{n}{p}}\|f\|_{L^p(B(x,t))} \tag{1}$$

holds for all $B(x,r)$ *and for all* $f\in L^p_{loc}\left(R^n\right)$.

Moreover if $p=1$, *then the inequality*

$$\|Mf\|_{WL^1(B(x,r))}\lesssim r^n\sup_{t>2r}t^{-n}\|f\|_{L^1(B(x,t))}$$

holds for all $B(x,r)$ *and for all* $f\in L^1_{loc}\left(R^n\right)$.

Theorem 1. *[9] 1. If* $f\in\widetilde{L}^{1,\lambda}\left(R^n\right)$, $0\le\lambda<n$, *then* $Mf\in W\widetilde{L}^{1,\lambda}\left(R^n\right)$ *and*

$$\|Mf\|_{W\widetilde{L}^{1,\lambda}}\le C_{1,\lambda,\mu}\|f\|_{\widetilde{L}^{1,\lambda}}, \tag{2}$$

where $C_{1,\lambda,\mu}$ *is independent of* f.

2. If $f\in\widetilde{L}^{p,\lambda}\left(R^n\right)$, $1<p<\infty$, $0\le\lambda<n$, *then* $Mf\in\widetilde{L}^{p,\lambda}\left(R^n\right)$ *and*

$$\|Mf\|_{\widetilde{L}^{p,\lambda}}\le C_{p,\lambda,\mu}\|f\|_{\widetilde{L}^{p,\lambda}} \tag{3}$$

where $C_{p,\lambda,\mu}$ *depends only on* p,λ,μ *and* n.

Lemma 3. *[16] Let* p *be the harmonic mean of* $p_1,\ldots,p_m>1$ *and* $\vec{f}\in L^1_{loc}\left(R^n\right)\times\times\ldots\times L^1_{loc}\left(R^n\right)$. *Then there exists a constant* $C>0$ *such that for any* $x\in R^n$

$$\mathcal{M}\left(\vec{f}\right)(x) \le \prod_{j=1}^{m}\left[M\left(f_j^{\frac{p_j}{p}}\right)(x)\right]^{\frac{p}{p_j}}. \tag{4}$$

When $m \ge 2$, we find out $\mathcal{M}$ also have the same properties by providing the following multi-version of the Theorem 1.

Theorem 2. *[15] Let p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m}\frac{\lambda_j}{p_j} \quad \text{for} \quad 0 \le \lambda_j < n. \tag{5}$$

(i) If $p > 1$, then the operator $\mathcal{M}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda}(R^n)\times\ldots\times\widetilde{L}^{p_m,\lambda}(R^n)$ to modified Morrey space $\widetilde{L}^{p,\lambda}(R^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in \widetilde{L}^{p_1,\lambda}(R^n)\times\ldots\times\widetilde{L}^{p_m,\lambda}(R^n)$

$$\| \mathcal{M}\vec{f} \|_{\widetilde{L}^{p,\lambda}} \le C\prod_{j=1}^{m}\left\|f_j\right\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p = 1$, then the operator $\mathcal{M}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda}(R^n)\times\ldots\times\widetilde{L}^{p_m,\lambda}(R^n)$ to weak modified Morrey space $W\widetilde{L}^{p,\lambda}(R^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in \widetilde{L}^{p_1,\lambda}(R^n)\times\ldots\times\widetilde{L}^{p_m,\lambda}(R^n)$

$$\| \mathcal{M}\vec{f} \|_{W\widetilde{L}^{p,\lambda}} \le C\prod_{j=1}^{m}\left\|f_j\right\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

3. Multi-sublinear maximal commutator operator on product total Morrey spaces

In this section we find necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator $\mathcal{M}_{\vec{b}}$ from $\vec{f} \in \widetilde{L}^{p_1,\lambda}(R^n)\times\ldots\times\widetilde{L}^{p_m,\lambda}(R^n)$ to $\widetilde{L}^{p,\lambda}(R^n)$ spaces.

Theorem 3. *[18, Lemma 1] If $b \in BMO(G)$, then for any $q \in (0,1)$, there exists a positive constant C such that*

$$M_q^{\#}(M_b f)(x) \le C\|b\|_* M^2 f(x)$$

for every $x \in G$ and for all $f \in L^1_{loc}(G)$.

Lemma 4. *If $\vec{b} \in BMO^m(R^n)$, then there exists a positive constant C such that*

$$\mathcal{M}_{\vec{b}}\left(\vec{f}\right)(x) \lesssim \sum_{k=1}^{m}\|b_k\|_{BMO} M^2 f_i(x)\prod_{j\ne i} M f_j(x)$$

for almost every $x \in R^n$ *and for all* $\vec{f} \in L^1_{loc}(R^n) \times \ldots \times L^1_{loc}(R^n)$.

Proof. Note that

$$\mathcal{M}_{b_i}\left(\vec{f}\right)(x) \lesssim M_{b_i} f_i(x) \prod_{j \neq i} M f_j(x)$$

and [18, Lemma 1]

$$M_{b_i} f_i(x) \lesssim \|b_i\|_{BMO} M^2 f_i(x) \prod_{j \neq i} M f_j(x),$$

then

$$\mathcal{M}_{\vec{b}}\left(\vec{f}\right)(x) = \sum_{i=1}^{m} \mathcal{M}_{b_i}\left(\vec{f}\right)(x) \leq \sum_{i=1}^{m} M_{b_i} f_i(x) \prod_{j \neq i} M f_j(x)$$

$$\lesssim \sum_{i=1}^{m} \|b_i\|_{BMO} M^2 f_i(x) \prod_{j \neq i} M f_j(x).$$

Theorem 4. *Let* b_i *be a real valued, locally integrable function in* R^n, $1 < p_i < \infty$, $0 \leq \lambda_i \leq n$ *with* $i = 1, \ldots, m$. *Let also* $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$, $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \ldots + \frac{\lambda_m}{p_m}$. *Then the following assertions are equivalent:*

The following assertions are equivalent:

(i) $\vec{b} \in BMO^m(R^n)$.

(ii) *The operator* $\mathcal{M}_{\vec{b}}$ *is bounded from* $\widetilde{L}^{p_1,\lambda}(R^n) \times \ldots \times \widetilde{L}^{p_m,\lambda}(R^n)$ *to* $\widetilde{L}^{p,\lambda}(R^n)$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $\vec{b} \in BMO^m(R^n)$. Combining Theorem 2 and Lemma 4, applying Hölder's inequality (see [5, Exercise 1.1.2, p. 10]), we get

$$\| \mathcal{M}_{\vec{b}}(\vec{f}) \|_{\widetilde{L}^{p,\lambda}} = \sup_{x \in R^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \| \mathcal{M}_{\vec{b}}\left(\vec{f}\right) \|_{L^p(B(x,t))}$$

$$\leq \sup_{x \in R^n, t > 0} [t]_1^{-\frac{\lambda}{p}} \sum_{i=1}^{m} \|b_i\|_{BMO} \left\| M^2 f_i \prod_{j \neq i} M f_j \right\|_{L^p(B(x,t))}$$

$$\leq \sup_{x \in R^n, t > 0} [t]_1^{-\frac{\lambda}{p}} \sum_{i=1}^{m} \|b_i\|_{BMO} \left\| M^2 f_i \right\|_{L^{p_i}(B(x,t))} \prod_{j \neq i} \left\| M f_j \right\|_{L^{p_j}(B(x,t))}.$$

From the inequality (1) we get

$$\left\| M^2 f_i \right\|_{L^{p_i}(B(x,t))} \lesssim t^{\frac{n}{p_i}} \sup_{\tau > 2t} \tau^{-\frac{n}{p_i}} \left\| M f_i \right\|_{L^p(B(x,\tau))}$$

$$\lesssim t^{\frac{n}{p_i}} \sup_{\tau>2t} \tau^{-\frac{n}{p_i}} \tau^{\frac{n}{p_i}} \sup_{\rho>2\tau} \rho^{-\frac{n}{p_i}} \left\| M f_i \right\|_{L^{p_j}(B(x,\rho))}$$

$$= t^{\frac{n}{p_i}} \sup_{\tau>2t} \sup_{\rho>2\tau} \rho^{-\frac{n}{p_i}} \| f_i \|_{L^{p_i}(B(x,\rho))} \le t^{\frac{n}{p_i}} \sup_{\rho>4\tau} \rho^{-\frac{n}{p_i}} \| f_i \|_{L^{p_i}(B(x,\rho))}$$

$$\le \| f_i \|_{\widetilde{L}^{p_i,\lambda_i}} t^{\frac{n}{p_i}} \sup_{\rho>4t} \rho^{-\frac{n}{p_i}} [\rho]_1^{\frac{\lambda_i}{p_i}} = \| f_i \|_{\widetilde{L}^{p_i,\lambda_i}} t^{\frac{n}{p_i}} \sup_{\rho>4t} [\rho]_1^{\frac{\lambda_i - n}{p_i}}$$

$$\le \| f_i \|_{\widetilde{L}^{p_i,\lambda_i}} [t]_1^{\frac{\lambda_i}{p_i}}$$

and

$$\left\| M f_j \right\|_{L^{p_j}(B(x,t))} \lesssim t^{\frac{n}{p_j}} \sup_{\tau>2t} \tau^{-\frac{n}{p_i}} \| f_j \|_{L^{p_j}(B(x,\tau))}$$

$$\le \| f_i \|_{\widetilde{L}^{p_i,\lambda_i}} t^{\frac{n}{p_i}} \sup_{\rho>4t} \rho^{-\frac{n}{p_i}} [\rho]_1^{\frac{\lambda_j}{p_j}} = \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} t^{\frac{n}{p_j}} \sup_{\rho>4\tau} [\rho]_1^{\frac{\lambda_j - n}{p_i}}$$

$$\le \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} [t]_1^{\frac{\lambda_j}{p_j}} .$$

Then

$$\| \mathcal{M}_{\vec{b}}(\vec{f}) \|_{\widetilde{L}^{p,\lambda}} \le \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} \sum_{i=1}^m \|b_i\|_{BMO} \left\| M^2 f_i \right\|_{L^{p_i}(B(x,t))} \prod_{j\ne i} \left\| M f_j \right\|_{L^{p_j}(B(x,t))}$$

$$\le \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} \sum_{i=1}^m \|b_i\|_{BMO} \| f_i \|_{\widetilde{L}^{p_i,\lambda_i}} [t]_1^{\frac{\lambda_i}{p_i}} \prod_{j\ne i} \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} [t]_1^{\frac{\lambda_j}{p_j}}$$

$$\le \sup_{x\in R^n, t>0} [t]_1^{-\frac{\lambda}{p}} \sum_{i=1}^m \|b_i\|_{BMO} \prod_{j=1}^m \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} [t]_1^{\frac{\lambda_j}{p_j}}$$

$$\le \sum_{i=1}^m \|b_i\|_{BMO} \prod_{j=1}^m \| f_j \|_{\widetilde{L}^{p_j,\lambda_j}} .$$

$(ii) \Rightarrow (i)$. Assume that $\mathcal{M}_{\vec{b}}$ is bounded from $\widetilde{L}^{p_1,\lambda}(R^n) \times \ldots \times \widetilde{L}^{p_m,\lambda}(R^n)$ to $\widetilde{L}^{p,\lambda}(R^n)$. Let $B(x,t)$ be a fixed ball. We consider $\vec{f} = \vec{\chi}_B$, where $\vec{\chi}_B = \underbrace{\{\chi_B, \ldots, \chi_B\}}_{m\ times}$. It is easy to compute that

$$\|\chi_B\|_{\widetilde{L}^{p_i,\lambda_i}} \approx \sup_{y\in R^n, t>0} \left([t]_1^{-\lambda_1} \int_{B(y,t)} \chi_B(z) dz \right)^{\frac{1}{p_i}}$$

$$= \sup_{y\in R^n, t>0} \left(\left|B(y,t)\cap B\right| [t]_1^{-\lambda_i}\right)^{\frac{1}{p_i}}$$

$$= \sup_{B(y,t)\subseteq B} \left(\left|B(y,t)\right| [t]_1^{-\lambda_1}\right)^{\frac{1}{p_i}} = r^{\frac{n}{p_i}} [r]_1^{-\frac{\lambda_i}{p_i}} . \tag{6}$$

On the other hand, since

$$\mathcal{M}_{\vec{b}}(\vec{\chi}_B)(x) \gtrsim \sum_{i=1}^{m} \frac{1}{|B|} \int_B \left|b_i(z) - (b_i)_B\right| dz \quad \text{for all} \quad x \in B,$$

we have

$$\left\|\mathcal{M}_{\vec{b}}(\vec{\chi}_B)\right\|_{\tilde{L}^{p,\lambda}} \approx \sup_{B(y,t)} \left([t]_1^{-\lambda} \right) \int_{B(y,t)} \left|\mathcal{M}_{\vec{b}}(\vec{\chi}_B)(z)\right|^p dz \Big)^{\frac{1}{p}}$$

$$\gtrsim r^{\frac{n}{p}} [r]_1^{-\frac{\lambda}{p}} \sum_{i=1}^{m} \frac{1}{|B|} \int_B \left|b_i(z) - (b_i)_B\right| dz. \tag{7}$$

Since by assumption

$$\left\|\mathcal{M}_{\vec{b}}(\vec{\chi}_B)\right\|_{\tilde{L}^{p,\lambda}} \lesssim \prod_{j=1}^{m} \left\|\chi_B\right\|_{\tilde{L}^{p_j,\lambda_j}},$$

by (6) and (7), we get that

$$\sum_{i=1}^{m} \frac{1}{|B|} \int_B \left|b_i(z) - (b_i)_B\right| dz \lesssim 1.$$

4. Conclusion

We find necessary and sufficient conditions for the boundedness of the multi-sublinear maximal commutator operator $\mathcal{M}_{\vec{b}}$ from product modified Morrey space $\tilde{L}^{p_1,\lambda}(R^n)\times\ldots\times\tilde{L}^{p_m,\lambda}(R^n)$ to modified Morrey space $\tilde{L}^{p,\lambda}(R^n)$.

5. Acknowledgements

The author would like to express his gratitude to the referee for his/her very valuable comments and kindly suggestion.

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